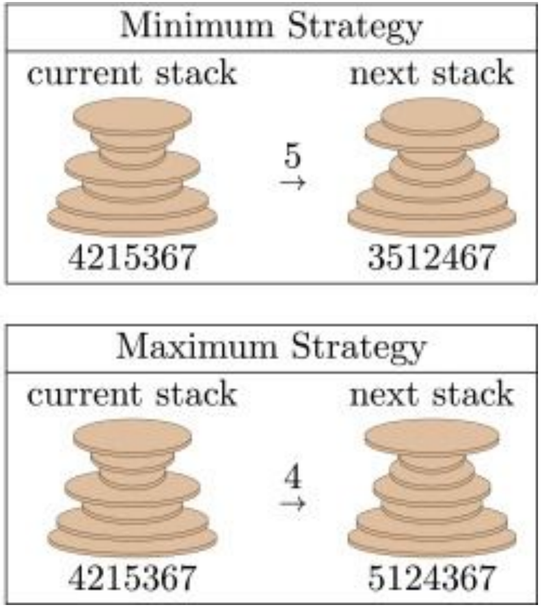
Comp 555 - BioAlgorithms - Spring 2021



- Problem set #6 is due on Thursday
- Final exam study session?

Genome Rearrangements - Continued

In search of Approximation Ratios

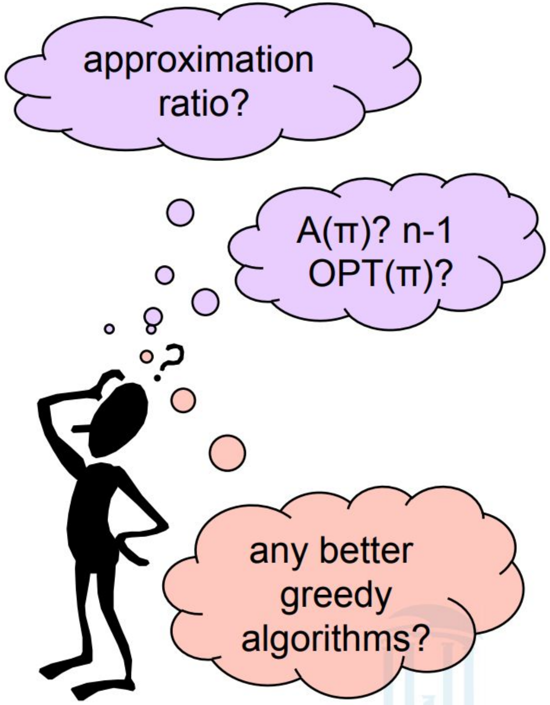
```
def GreedyReversalSort(pi):
    for i in range(len(pi)-1):
        j = pi.index(min(pi[i:]))
        if (j != i):
            pi = pi[:i]
               + [v for v in reversed(pi[i:j+1])]
               + pi[j+1:]
    return pi
```

A(Π)

Step 0: 6 1 2 3 4 5
Step 1: 1 6 2 3 4 5
Step 2: 1 2 6 3 4 5
Step 3: 1 2 3 6 4 5
Step 4: 1 2 3 4 6 5
Step 5: 1 2 3 4 5 6

OPT(Π)?

Step 0: 6 1 2 3 4 5
Step 1: 5 4 3 2 1 6
Step 2: 1 2 3 4 5 6



New Idea: Adjacencies

- Adjacencies are locally sorted runs.
- Assume a permutation:

$$\Pi = \pi_1, \pi_2, \pi_3, \ldots \pi_{n-1}, \pi_n$$

- A pair of neighboring elements π_i and π_{i+1} are *adjacent* if:

$$\pi_{i+1} = \pi_i \pm 1$$

- For example:

$$\Pi = 1, 9, \underline{3, 4}, \underline{7, 8}, 2, \underline{6, 5}$$

- (3,4) and (7,8) and (6,5) are adjacencies.

Adjacencies and Breakpoints

- *Breakpoints* occur between neighboring non-adjacent elements

$$\Pi = 1, \mid 9, \mid \underline{3, 4}, \mid \underline{7, 8}, \mid 2, \mid \underline{6, 5}$$

- There are 5 breakpoints in our permutation between pairs (1,9), (9,3), (4,7), (8,2) and (2,5)
- We define b(Π) as the number of breakpoints in permutation Π

Extending Permutations

- One can place two special elements, $\pi_0 = 0$ and $\pi_{n+1} = n+1$ at the beginning and end of Π respectively

$$1, \mid 9, \mid \underline{3, 4,} \mid \underline{7, 8,} \mid 2, \mid \underline{6, 5}$$

$$\downarrow$$

$$\Pi = 0\ 1, \mid 9, \mid \underline{3, 4,} \mid \underline{7, 8,} \mid 2, \mid \underline{6, 5,} \mid 10$$

- An additional breakpoint was created after extending
- An extended permutation of length n can have at most (n+1) breakpoints
- (n−1) between the original elements plus 2 for the extending elements

How Reversals Effect Breakpoints

- Breakpoints are the *targets* for sorting by reversals.
- Once they are removed, the permutation is sorted.
- Each "useful" reversal eliminates at least 1, and at most 2 breakpoints.
- Consider the following application of GreedyReversalSort(Extend(Π))

$$\Pi = \quad 2, 3, 1, 4, 6, 5$$

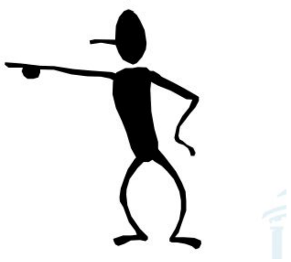
$$0 \mid \overline{2, 3 \mid 1} \mid 4 \mid 6, 5 \mid 7 \qquad b(\Pi) = 5$$

$$0, \; 1 \mid \overline{3, 2} \mid 4 \mid 6, 5 \mid 7 \qquad b(\Pi) = 4$$

$$0, 1, 2, 3, 4 \mid \overline{6, 5} \mid 7 \qquad b(\Pi) = 2$$

$$0, 1, 2, 3, 4, 5, 6, 7 \qquad b(\Pi) = 0$$

$$\textit{required reversals} \geq \frac{b(\pi)}{2}$$



Sorting-by-Reversals: A second Greedy Algorithm

BreakpointReversalSort(π):

1. while $b(\pi) > 0$:
2. Among all possible reversals, choose reversal ρ minimizing $b(\pi)$
3. $\Pi \leftarrow \Pi \cdot \rho(i, j)$
4. output Π
5. return

The "greedy" concept here is to reduce as many breakpoints as possible at each step.

Does it always terminate?

What if no reversal reduces the number of breakpoints?

0 1 2 | 5 6 7 | 3 4 | 8 9

Yet Another New Idea: *Strips*

Strip: an interval between two consecutive breakpoints in a permutation

- *Decreasing strip:* strip of elements in decreasing order (e.g. 6 5 and 3 2).
- *Increasing strip:* strip of elements in increasing order (e.g. 7 8)
- A single-element strip can be declared either increasing or decreasing.
- We will choose to declare them as *decreasing* with exception of extension strips (with 0 and n+1)

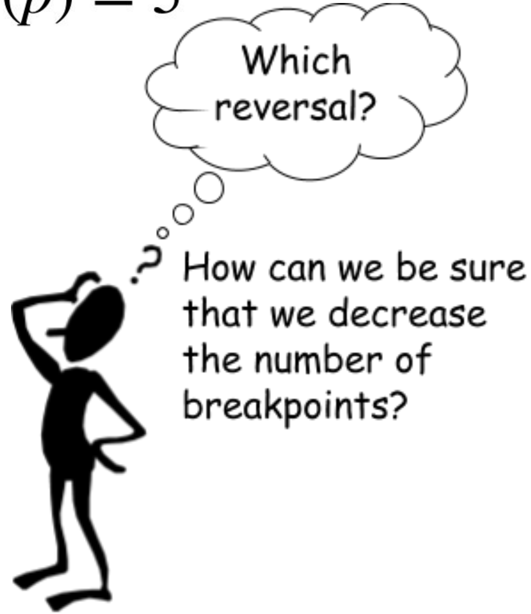
$$\overrightarrow{0, 1}, \overleftarrow{9}, \overleftarrow{4, 3}, \overrightarrow{7, 8}, \overleftarrow{2}, \overrightarrow{5, 6}, \overrightarrow{10}$$

Reducing the Number of Breakpoints

- Consider Π =1,4,6,5,7,8,3,2

$$\overrightarrow{0, 1}, \mid \overleftarrow{4}, \mid \overleftarrow{6, 5}, \mid \overrightarrow{7, 8}, \mid \overleftarrow{3, 2}, \mid \overrightarrow{9} \qquad b(p) = 5$$

If permutation p contains at least one decreasing strip, then there exists a reversal r which decreases the number of breakpoints (i.e. $b(p \bullet r) < b(p)$).



Things to Consider

- Consider Π = 1,4,6,5,7,8,3,2

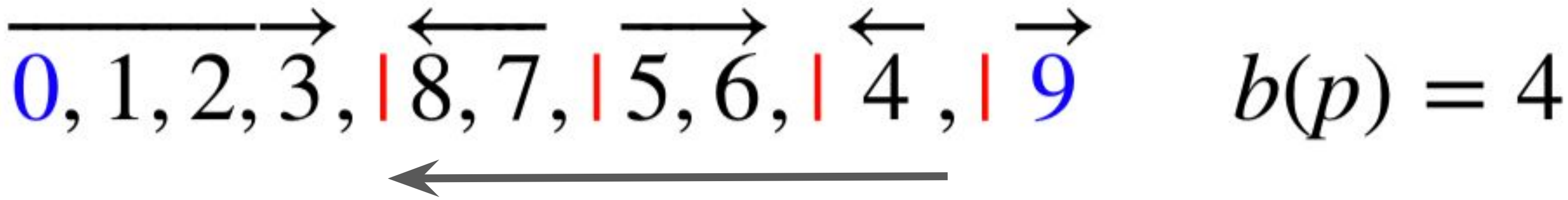
$$\overrightarrow{0,1,} \mid \overleftarrow{4,} \mid \overleftarrow{6,5,} \mid \overrightarrow{7,8,} \mid \overleftarrow{3,2,} \mid \overrightarrow{9} \qquad b(p) = 5$$

- Choose the decreasing strip with the smallest element k in Π
  - It'll always be the right-most element of that strip
- Find k−1 in the permutation
  - it'll always be flanked by a breakpoint
- Reverse the segment between k and k−1

1

Things to Consider

- Consider Π = 1,4,6,5,7,8,3,2



- Choose the decreasing strip with the smallest element k in Π
  - It'll always be the right-most element of that strip
- Find k−1 in the permutation
  - it'll always be flanked by a breakpoint
- Reverse the segment between k and k−1

2

Things to Consider

- Consider Π = 1,4,6,5,7,8,3,2

$$0, 1, 2, 3, 4, \mid 6, 5, \mid 7, 8, 9 \qquad b(p) = 2$$

- Choose the decreasing strip with the smallest element k in Π
  - It'll always be the right-most element of that strip
- Find k−1 in the permutation
  - it'll always be flanked by a breakpoint
- Reverse the segment between k and k−1

3

Things to Consider

- Consider Π = 1,4,6,5,7,8,3,2

$$\overrightarrow{0, 1, 2, 3, 4, 5, 6, 7, 8, 9} \qquad b(p) = 0$$

- Choose the decreasing strip with the smallest element k in Π
  - It'll always be the right-most element of that strip
- Find k−1 in the permutation
  - it'll always be flanked by a breakpoint
- Reverse the segment between k and k−1

Done

Things to Consider

- Consider Π = 1,4,6,5,7,8,3,2

$$\overrightarrow{0,1},\ |\ \overleftarrow{4},\ |\ \overleftarrow{6,5},\ |\ \overrightarrow{7,8},\ |\ \overleftarrow{3,2},\ |\ \overrightarrow{9} \qquad b(p) = 5$$

$$\overrightarrow{0,1,2,3},\ |\ \overleftarrow{8,7},\ |\ \overrightarrow{5,6},\ |\ \overleftarrow{4},\ |\ \overrightarrow{9} \qquad b(p) = 4$$

$$\overrightarrow{0,1,2,3,4},\ |\ \overleftarrow{6,5},\ |\ \overrightarrow{7,8,9} \qquad b(p) = 2$$

$$\overrightarrow{0,1,2,3,4,5,6,7,8,9} \qquad b(p) = 0$$

$$d(\Pi) = 3$$

Any other way?

- Reconsider Π = 1,4,6,5,7,8,3,2

0, 1, | 4, | 6, 5, | 7, 8, | 3, 2, | 9 $\qquad b(p) = 5$

0, 1, | 4, 5, 6, 7, 8, | 3, 2, | 9 $\qquad b(p) = 3$

0, 1, | 8, 7, 6, 5, 4, 3, 2, | 9 $\qquad b(p) = 2$

0, 1, 2, 3, 4, 5, 6, 7, 8, 9 $\qquad b(p) = 0$

What is the most possible breakpoints that can be removed in a single reversal?

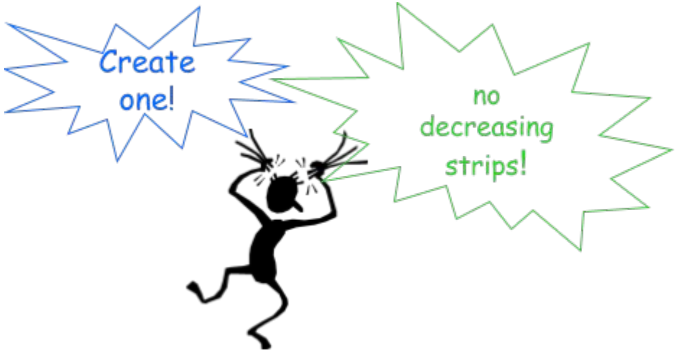
What is fewest number of reversals that could reduce 5 breakpoints to 0?

How many breakpoints are eliminated in the last reversal?

Potential Gotcha

$$\overrightarrow{0,1,2},\,|\,\overrightarrow{5,6,7},\,|\,\overrightarrow{3,4},\,|\,\overrightarrow{8,9} \qquad b(p) = 3$$

- If there is no decreasing strip, there may be *no strip-reversal* ρρ *that reduces the number of breakpoints* (i.e. $b(\Pi \dot{} \rho(i,j)) \geq b(\Pi)$ for any reversal ρ).
- However, reversing an increasing strip creates a decreasing strip, and the number of breakpoints remains unchanged.
- Then the number of breakpoints will be reduced in the following steps.



Potential Gotcha

$$\overrightarrow{0,1,2,}\,|\,\overrightarrow{5,6,7,}\,|\,\overrightarrow{3,4,}\,|\,\overrightarrow{8,9} \qquad b(p)=3$$

$$\overrightarrow{0,1,2,}\,|\,\overleftarrow{7,6,5,}\,|\,\overrightarrow{3,4,}\,|\,\overrightarrow{8,9} \qquad b(p)=3$$

- If there is no decreasing strip, there may be *no strip-reversal* ρρ *that reduces the number of breakpoints* (i.e. b(Π˙ρ(i,j)) ≥ b(Π) for any reversal ρ).
- However, reversing an increasing strip creates a decreasing strip, and the number of breakpoints remains unchanged.
- Then the number of breakpoints will be reduced in the following steps.

Putting it all together

1. With each reversal, one can remove at most 2 breakpoints
2. If there is any *decreasing* strip there exists a reversal that will remove at least one breakpoint
3. If breakpoints remain and there is no *decreasing* strip one can be created by reserving *any* remaining strip

$\overrightarrow{0,1,2,} \mid \overrightarrow{5,6,7,} \mid \overrightarrow{3,4,} \mid \overrightarrow{8,9}$	$b(p) = 3$	$\rho(3,5)$
$\overrightarrow{0,1,2,} \mid \overleftarrow{7,6,5,} \mid \overrightarrow{3,4,} \mid \overrightarrow{8,9}$	$b(p) = 3$	$\rho(6,7)$
$\overrightarrow{0,1,2,} \mid \overleftarrow{7,6,5,4,3,} \mid \overrightarrow{8,9}$	$b(p) = 2$	$\rho(3,7)$
$\overrightarrow{0,1,2,3,4,5,6,7,8,9}$	$b(p) = 0$	*Done!*

An optimal algorithm would remove 2 breakpoints at every step. The last reversal always removes 2 breakpoints, thus if the number of breakpoints is odd, even the optimal algorithm must make at least one reversal that removes only 1 breakpoint.

An Improved Breakpoint Reversal Sort

ImprovedBreakpointReversalSort(π)

```
while b(π) > 0
    if π has a decreasing strip
        Among all possible reversals, choose reversal ρ that minimizes b(π • ρ)
    else
        Choose a reversal ρ that flips an increasing strip in π
    π ← π • ρ
output π
return
```

Breakpoints and Strips

```
In [11]: def hasBreakpoints(seq):
             """ returns True if sequences is not strictly increasing by 1 """
             for i in range(1, len(seq)):
                 if (seq[i] != seq[i-1] + 1):
                     return True
             return False

         def getStrips(seq):
             """ find contained intervals where sequence is ordered, and return intervals
             in as lists, increasing and decreasing. Single elements are considered
             decreasing. "Contained" excludes the first and last interval. """
             deltas = [seq[i+1] - seq[i] for i in range(len(seq)-1)]
             increasing = list()
             decreasing = list()
             start = 0
             for i, diff in enumerate(deltas):
                 if (abs(diff) == 1) and (diff == deltas[start]):
                     continue
                 if (start > 0):
                     if deltas[start] == 1:
                         increasing.append((start, i+1))
                     else:
                         decreasing.append((start, i+1))
                 start = i+1
             return increasing, decreasing
```

Handle Reversals

In [15]:

```
def pickReversal(seq, strips):
    """ test each decreasing interval to see if it leads to a reversal that
    removes two breakpoints, otherwise, return a reversal that removes only one """
    for i, j in strips:
        k = seq.index(seq[j-1]-1)
        if (seq[k+1] + 1 == seq[j]):
            # removes 2 breakpoints
            return 2, (min(k+1, j), max(k+1, j))
    # In the worst case we remove only one, but avoid the length "1" strips
    for i, j in strips:
        k = seq.index(seq[j-1]-1)
        if (j - i > 1):
            break
    return 1, (min(k+1, j), max(k+1, j))

def doReversal(seq,reversal):
    i, j = reversal
    return seq[:i] + [element for element in reversed(seq[i:j])] + seq[j:]
```

Let's do it!

```
In [13]: def improvedBreakpointReversalSort(seq, verbose=True):
             seq = [0] + seq + [max(seq)+1]                          # Extend sequence
             N = 0
             while hasBreakpoints(seq):
                 increasing, decreasing = getStrips(seq)
                 if len(decreasing) > 0:                              # pick a reversal that removes a decreasing strip
                     removed, reversal = pickReversal(seq, decreasing)
                 else:
                     removed, reversal = 0, increasing[0]             # No breakpoints can be removed
                 if verbose:
                     print("Strips:", increasing, decreasing)
                     print("%d: %s  rho%s" % (removed, seq, reversal))
                     input("Press Enter:")
                 seq = doReversal(seq,reversal)
                 N += 1
             if verbose:
                 print(seq, "Sorted")
             return N

         # Also try: [1,9,3,4,7,8,2,6,5]
         print(improvedBreakpointReversalSort([3,4,1,2,5,6,7,10,9,8], verbose=True))
```

```
Strips: [(1, 3), (3, 5), (5, 8)] [(8, 11)]
2: [0, 3, 4, 1, 2, 5, 6, 7, 10, 9, 8, 11]  rho(8, 11)
Press Enter:
Strips: [(1, 3), (3, 5)] []
0: [0, 3, 4, 1, 2, 5, 6, 7, 8, 9, 10, 11]  rho(1, 3)
Press Enter:
Strips: [(3, 5)] [(1, 3)]
1: [0, 4, 3, 1, 2, 5, 6, 7, 8, 9, 10, 11]  rho(3, 5)
Press Enter:
Strips: [] [(1, 5)]
2: [0, 4, 3, 2, 1, 5, 6, 7, 8, 9, 10, 11]  rho(1, 5)
Press Enter:
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] Sorted
4
```

Performance

- *ImprovedBreakPointReversalSort* is a greedy algorithm with a performance guarantee of no worse than 4 compared to an optimal algorithm
  - It eliminates at least one breakpoint in every two steps (flip an increasing then remove 1)
  - That's at most: 2b(Π) steps
  - An optimal algorithm could *at most* remove 2 breakpoints in every step, thus requiring b(Π)/2 steps
  - The approximation ratio is:

$$\frac{\mathcal{A}(\Pi)}{OPT(\Pi)} = \frac{2b(\Pi)}{\frac{b(\Pi)}{2}} = 4$$

- But there is a solution with far fewer flips

A Better Approximation Ratio

- If there is a decreasing strip, the next reversal reduces b(π) by at least one.
- The only bad case is when there is no decreasing strip.
  Then we do a reversal that does not reduce b(π).
- If we always choose a reversal reducing b(π) and, *at the same time, select a permutation such that the result has at least one decreasing strip*, the bad case would never occur.
- If all possible reversals that reduce b(π) create a permutation without decreasing strips, then there exists a reversal that reduces b(π) by 2 (Proof not given)!
- When the algorithm creates a permutation without a decreasing strip, the previous reversal must have reduced b(π) by two.
- At most b(π) reversals are needed.
- The improved Approximation ratio:

$$\frac{\mathcal{A}_{new}(\Pi)}{OPT(\Pi)} = \frac{b(\Pi)}{\frac{b(\Pi)}{2}} = 2$$

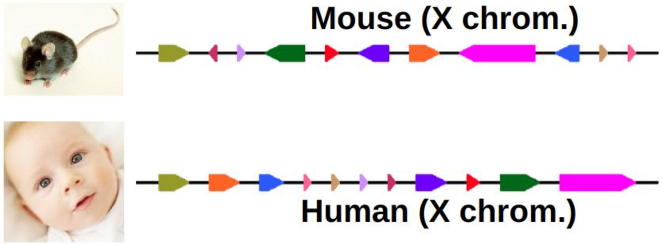
Comparing Greedy Algorithms

SimpleReversalSort

- Attempts to extend the prefix(π) at each step
- Approximation ratio (n−1)/(b(Π)/2) can be large

ImprovedBreakpointReversalSort

- Attempts to reduce the number of breakpoints at each step
- Approximation ratio b(Π)/(b(Π)/2) = 2x



Next Time

Randomized Algorithms