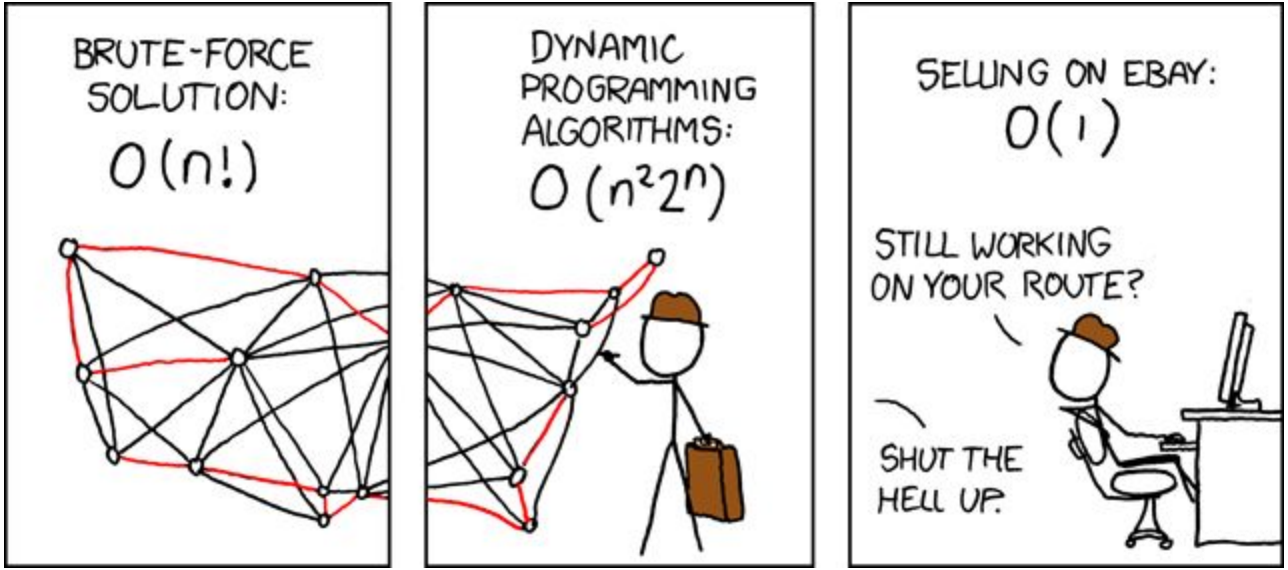
Comp 555 - BioAlgorithms - Spring 2020

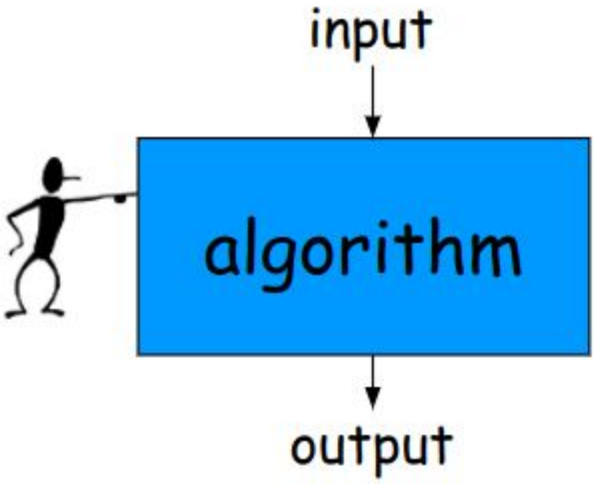


- Problem set #3 is due next Tuesday
- Midterm is set for next Thursday

Adventures in Dynamic Programming

An aside… what is an Algorithm?

An algorithm is a sequence of instructions that solves a well-formulated problem.

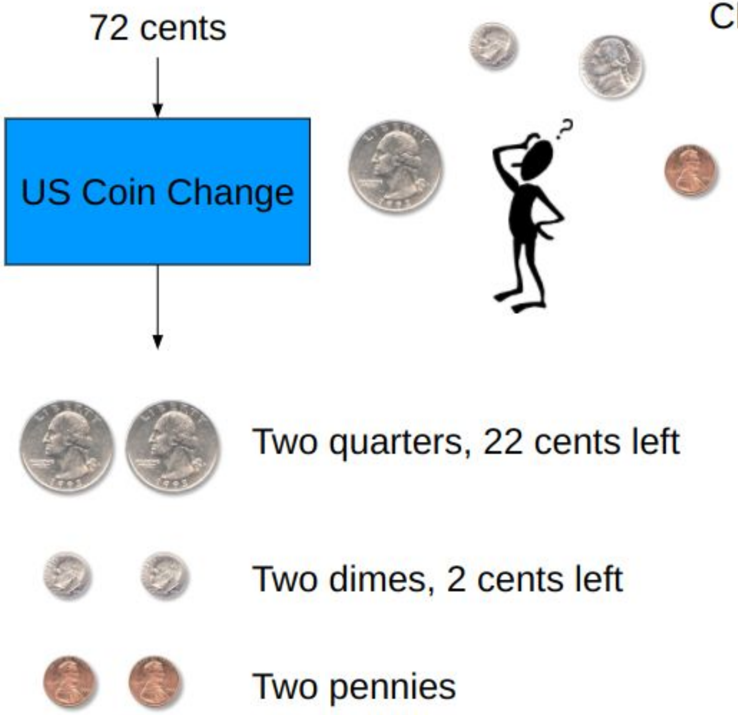


Algorithm:
Complexity
Correctness

Correctness

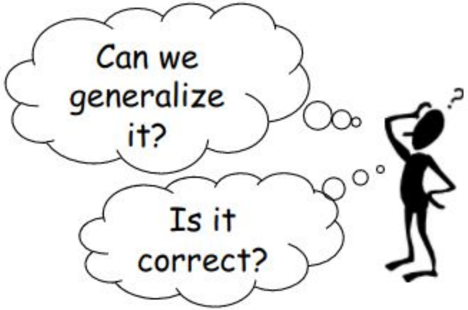
- An algorithm is correct only if it produces correct result for every valid input instance
    - An algorithm is incorrect answer if it cannot produce a correct result for one or more input instances,
- Coin change problem
    - **Input:** an amount of money *M* in cents, and a list of coin denominations $[c_1, c_2, \ldots, c_n]$
    - **Output:** the smallest number of coins that add to *M* (may not be unique)
- US coin change problem

US Coin Change



Classic Algorithm

$$
\begin{aligned}
& r \leftarrow M \\
& q \leftarrow r / 25 \\
& r \leftarrow r - 25 \cdot q \\
& d \leftarrow r / 10 \\
& r \leftarrow r - 10 \cdot d \\
& n \leftarrow r / 5 \\
& r \leftarrow r - 5 \cdot n \\
& p \leftarrow r
\end{aligned}
$$



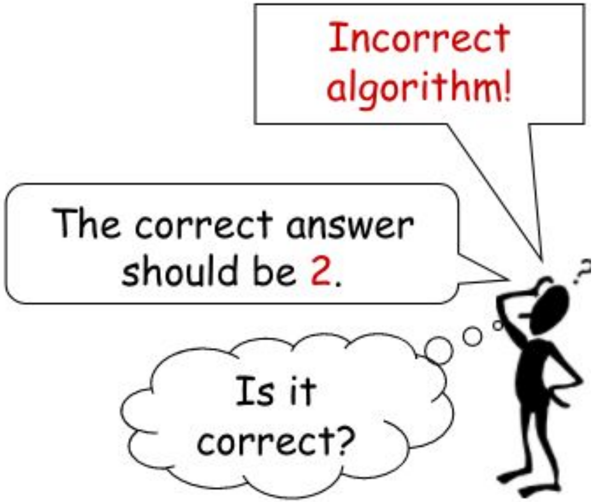
Change Problem

- Input:
  - an amount of money M
  - an array of denominations c = (c1, c2, …,cd) in order of decreasing value
- Output: the smallest number of coins

$M = 40$
$c = (25, 20, 10, 5, 1)$

$r \leftarrow M$
$n \leftarrow 0$
for $k \leftarrow 1$ to d
  $i_k \leftarrow r / c_k$
  $n \leftarrow n + i_k$
  $r \leftarrow r - c_k \times i_k$
return n

→ 3



A "Greedy" change approach

- **Key idea: Use as many of the largest available coin denomination so long as the sum is less than or equal to the change amount**

In [3]:

```python
def greedyChange(amount, denominations):
    # Goal is to produce the fewest coins to achieve
    # given target "amount"
    # Strategy: Give as many of the largest coin
    # denomination that is less than amount.
    solution = []
    for coin in denominations:
        i = amount // coin        # truncating integer divide
        solution.append(i)
        amount -= coin * i
    return solution

s1 = greedyChange(72, [25,10,5,1])
print(s1, sum(s1))
s2 = greedyChange(40, [25,10,5,1])
print(s2, sum(s2))
s3 = greedyChange(40, [25,20,10,5,1])
print(s3, sum(s3))
```

```
[2, 2, 0, 2] 6
[1, 1, 1, 0] 3
[1, 0, 1, 1, 0] 3
```

Another Approach?

- Let's bring back brute force
- Test every coin combination (where each denomination is less than 100) to see if it adds up to our target
- Is there exhaustive search algorithm?

In [8]:
```python
def exhaustiveChange(amount, denominations):
    bestN = 100
    count = [0 for i in range(len(denominations))]
    while True:
        for i, coinValue in enumerate(denominations):
            count[i] += 1
            if (count[i]*coinValue < 100):
                break
            count[i] = 0
        n = sum(count)
        if n == 0:
            break
        value = sum([count[i]*denominations[i] for i in range(len(denominations))])
        if (value == amount):
            if (n < bestN):
                solution = [count[i] for i in range(len(denominations))]
                bestN = n
    return solution

%time print(exhaustiveChange(40,[25,20,10,5,1]))
```

```
[0, 2, 0, 0, 0]
CPU times: user 688 ms, sys: 0 ns, total: 688 ms
Wall time: 672 ms
```

[0,1,2,3]	25
[0,1,2,3,4]	20
[0,...,9]	10
[0,...,19]	5
[0,...,99]	100

4*5*10*20*100 = 400000

Correct, but costly

- Our algorithm now gets the right answer for every value 1..100
- It must, because it considers every possible answer (that’s the good thing about brute force)
- There is a downside though

In [16]:
```
%time print(exhaustiveChange(40, [25,10,5,1]))
%time print(exhaustiveChange(40, [25,20,10,5,1]))
%time print(exhaustiveChange(40, [13,11,7,5,3,1]))
```

```
[1, 1, 1, 0]
CPU times: user 155 ms, sys: 0 ns, total: 155 ms
Wall time: 149 ms
[0, 2, 0, 0, 0]
CPU times: user 632 ms, sys: 0 ns, total: 632 ms
Wall time: 628 ms
[0, 3, 1, 0, 0, 0]
CPU times: user 2min 50s, sys: 0 ns, total: 2min 50s
Wall time: 2min 50s
```

Other tricks?

A Branch-and-bound algorithm, almost identical to brute force

In [17]:
```python
def branchAndBoundChange(amount, denominations):
    bestN = amount
    count = [0 for i in range(len(denominations))]
    while True:
        for i, coinValue in enumerate(denominations):
            count[i] += 1
            if (count[i]*coinValue < amount):                # Set upper bound to amount rather than 100
                break
            count[i] = 0
        n = sum(count)
        if n == 0:
            break
        if (n > bestN):                                      # don't compute the amount if there are too many coins
            continue
        value = sum([count[i]*denominations[i] for i in range(len(denominations))])
        if (value == amount):
            if (n < bestN):
                solution = [count[i] for i in range(len(denominations))]
                bestN = n
    return solution

%time print(branchAndBoundChange(40, [13,11,7,5,3,1]))
```

```
[0, 3, 1, 0, 0, 0]
CPU times: user 317 ms, sys: 0 ns, total: 317 ms
Wall time: 299 ms
```

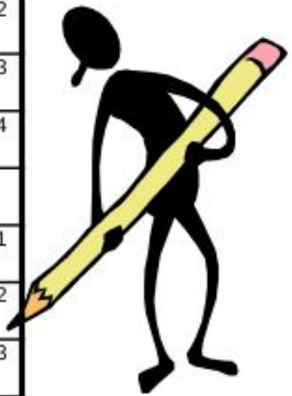
..Correct, and it works well for many cases, but can be as slow as an exhaustive search for some inputs (try 99).

Is there another Approach?

Tabulating Answers

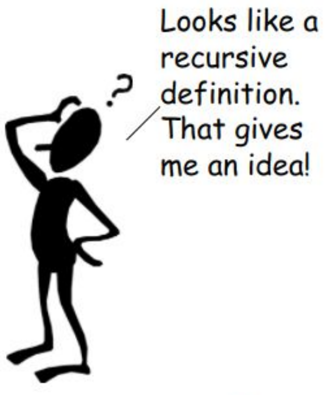
- If it is costly to compute the answer for a given input, then there may be advantages to caching the result of previous calculations in a table
- This trades-off time-complexity for space
- How could we fill in the table in the first place?
- Run our best correct algorithm
- Can the table itself be used to speed up the process?

Amt	25	20	10	5	1	Amt	25	20	10	5	1
1¢					1	42¢		2			2
2¢					2	43¢		2			3
3¢					3	44¢		2			4
4¢					4	45¢		2		1	
5¢				1		46¢		2		1	1
6¢				1	1	47¢		2		1	2
7¢				1	2	48¢		2		1	3
8¢				1	3	49¢		2		1	4
9¢				1	4	50¢	2				
10¢			1			51¢	2				1
11¢			1		1	52¢	2				2



Solutions using a Table

- Suppose you are asked to fill-in the unknown table entry for 67¢
- It must differ from a previously known optimal result by at most one coin...
- So what are the possibilities?
  - BestChange(67¢) = 25¢ + BestChange(42¢), or
  - BestChange(67¢) = 20¢ + BestChange(47¢), or
  - BestChange(67¢) = 10¢ + BestChange(57¢), or
  - BestChange(67¢) = 5¢ + BestChange(62¢), or
  - BestChange(67¢) = 1¢ + BestChange(66¢)



A Recursive Coin-Change Algorithm

In [23]:
```
def RecursiveChange(M, c):
    if (M == 0):
        return [0 for i in range(len(c))]
    smallestNumberOfCoins = M+1
    for i in range(len(c)):
        if (M >= c[i]):
            thisChange = RecursiveChange(M - c[i], c)
            thisChange[i] += 1
            if (sum(thisChange) < smallestNumberOfCoins):
                bestChange = thisChange
                smallestNumberOfCoins = sum(thisChange)
    return bestChange

%time print(RecursiveChange(40, [1,3,5,7,11,13]))
```

```
[1, 0, 0, 0, 0, 3]
CPU times: user 6min 43s, sys: 16 ms, total: 6min 43s
Wall time: 6min 43s
```

Oops… it got slower. Why?
(Not to mention, it found another “different” correct answer.)

Recursion Recalculations

- Recursion often results in many redundant calls
- Even after only two levels of recursion 6 different change values are repeated multiple times
- How can we avoid this repetition?
- Cache precomputed results in a table!

Change(40) = 25 + Change(15)
25 + 10 + Change(5)
25 + 5 + Change(10)
20 + Change(20)
20 + 20 + Change(0)
20 + 10 + Change(10)
20 + 5 + Change(15)
10 + Change(30)
10 + 25 + Change(5)
10 + 20 + Change(10)
10 + 10 + Change(20)
10 + 5 + Change(25)
5 + Change(35)
5 + 25 + Change(15)
5 + 20 + Change(10)
5 + 10 + Change(25)
5 + 5 + Change(30)

Back to Table Evaluation

- When do we fill in the values of our table?
- We could solve for change for every value from 1 up to M, thus we'd be gaurenteed to have found the best change for any value less than M when needed
- Thus, instead of just trying to find the minimal number of coins to change M cents, we attempt the solve the superficially harder problem of solving for the optimal change for all values from 1 to M

1¢ = [0,0,0,0,1]	2¢ = [0,0,0,0,2]	3¢ = [0,0,0,0,3]	. . .	M¢ = [?,?,?,?,?]

Change via Dynamic Programming

In [27]:

```
def DPChange(M, c):
    change = [[0 for i in range(len(c))]]
    for m in range(1,M+1):
        bestNumCoins = m+1
        for i in range(len(c)):
            if (m >= c[i]):
                thisChange = [x for x in change[m - c[i]]]
                thisChange[i] += 1
                if (sum(thisChange) < bestNumCoins):
                    change[m:m] = [thisChange]
                    bestNumCoins = sum(thisChange)
    return change[M]

%time print(DPChange(40, [1,3,5,7,11,13]))
%time print(DPChange(40, [1,3,5,7,11,13,17]))
%time print(DPChange(40, [1,3,5,7,11,13,17,19]))
```

```
[1, 0, 0, 0, 0, 3]
CPU times: user 3 ms, sys: 1e+03 µs, total: 4 ms
Wall time: 2.82 ms
[1, 0, 1, 0, 0, 0, 2]
CPU times: user 1e+03 µs, sys: 0 ns, total: 1e+03 µs
Wall time: 1.28 ms
[2, 0, 0, 0, 0, 0, 0, 2]
CPU times: user 0 ns, sys: 0 ns, total: 0 ns
Wall time: 462 µs
```

- BruteForceChange( ) was $O(d^M)$
- DPChange( ) is $O(Md)$

A Hybrid Approach: Memoization

- Often we can simply modify a recursive algorithm to “cache” the result of previous invocations
- FIll in table lazily as needed… as each call to progresses from M down to 1
- This “*lazy evaluated*” form of dynamic programming is often called “*Memoization*”

In [34]:

```
change = {}                                    # This is a cache for saving bestChange[M]

def MemoizedChange(M, c):
    global change
    if (M in change):                          # Check the cache first
        return [v for v in change[M]]
    if (len(change) == 0):                     # Initialize cache
        change[0] = [0 for i in range(len(c))]
    smallestNumberOfCoins = M+1
    for i in range(len(c)):
        if (M >= c[i]):
            thisChange = MemoizedChange(M - c[i], c)
            thisChange[i] += 1
            if (sum(thisChange) < smallestNumberOfCoins):
                bestChange = [v for v in thisChange]
                smallestNumberOfCoins = sum(thisChange)
    change[M] = [v for v in bestChange]        # Add new M to cache
    return bestChange

%time print(MemoizedChange(40, [1,3,5,7,11,13]))
```

```
[1, 0, 0, 0, 0, 3]
CPU times: user 541 µs, sys: 0 ns, total: 541 µs
Wall time: 477 µs
```

Dynamic Programming

- Dynamic Programming is a general technique for computing recurrence relations efficiently by storing partial or intermediate results
- Three keys to constructing a dynamic programming solution:
    1. Formulate the answer as a recurrence relation
    2. Consider all instances of the recurrence at each step
    3. Order evaluations so you will always have precomputed the needed partial results
- Memoization is an easy way to convert recursive solutions to a DP
- We'll see it again, and again

Next Time

- On to sequence alignment
- But first we'll learn how to navigate in Mathattan