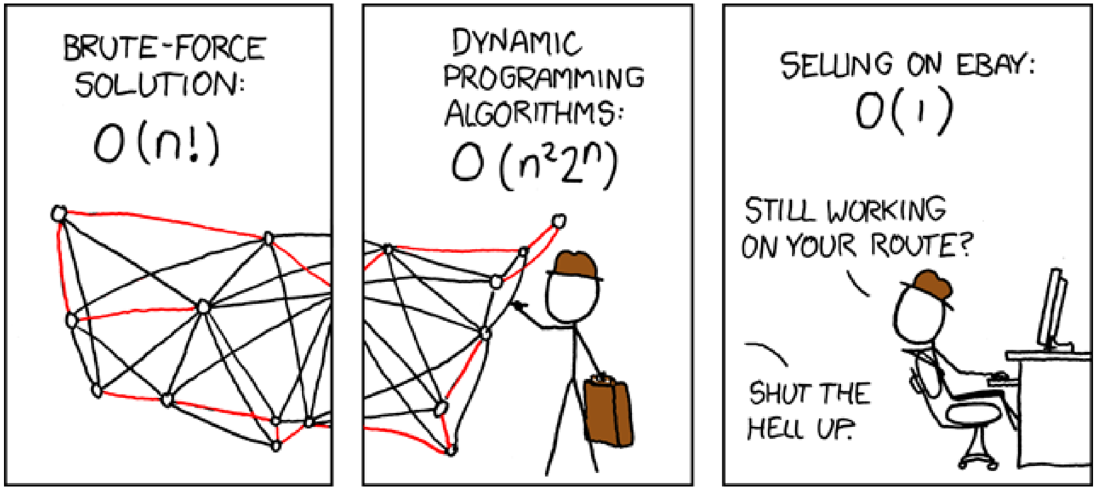
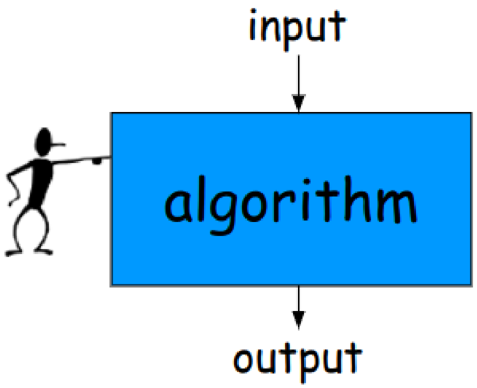
Returning to Dynamic Programming



What is an Algorithm?

- An algorithm is a sequence of instructions that one must perform in order to solve a well-formulated problem.



input

algorithm

output

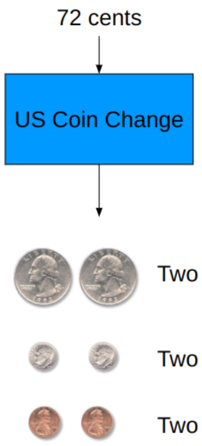
Algorithm:
Complexity
Correctness

Correctness

- An algorithm is correct only if it produces correct result for all input instances.
  - If the algorithm gives an incorrect answer for one or more input instances, it is an incorrect algorithm.
- Coin change problem
  - **Input:** an amount of money M in cents
  - **Output:** the smallest number of coins
- US coin change problem

First Previous Next Notebook

US Coin Change


72 cents

US Coin Change

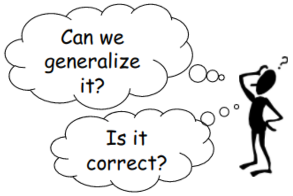
Two quarters, 22 cents left

Two dimes, 2 cents left

Two pennies

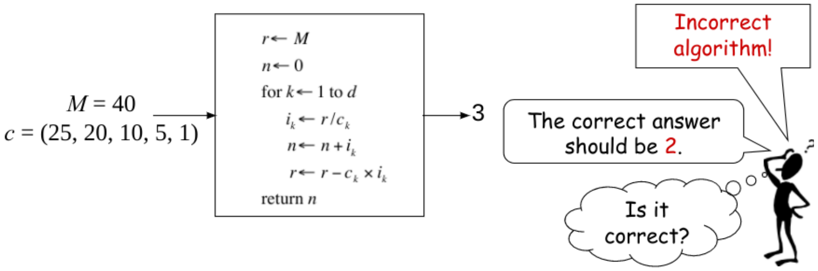
Classic Algorithm

```
$r \leftarrow M$
$q \leftarrow r/25$
$r \leftarrow r - 25 \cdot q$
$d \leftarrow r/10$
$r \leftarrow r - 10 \cdot d$
$n \leftarrow r/5$
$r \leftarrow r - 5 \cdot n$
$p \leftarrow r$
```



Change Problem

- Input:
    - an amount of money M
    - an array of denominations $c = (c_1, c_2, \ldots, c_d)$ in order of decreasing value
- Output: the smallest number of coins



```
$r \leftarrow M$
$n \leftarrow 0$
for $k \leftarrow 1$ to $d$
    $i_k \leftarrow r / c_k$
    $n \leftarrow n + i_k$
    $r \leftarrow r - c_k \times i_k$
return $n$
```

Another Approach?

- Let's bring back brute force
  - Test every coin combination and see if it adds up to our target
  - Is there exhaustive search algorithm?

```
def exhaustiveChange(amount, denominations):
    bestN = 100
    count = [0 for i in xrange(len(denominations))]
    while True:
        for i, coinValue in enumerate(denominations):
            count[i] += 1
            if (count[i]*coinValue < 100):
                break
            count[i] = 0
        n = sum(count)
        if n == 0:
            break
        value = sum([count[i]*denominations[i] for i in xrange(len(denominations))])
        if (value == amount):
            if (n < bestN):
                solution = [count[i] for i in xrange(len(denominations))]
                bestN = n
    return solution

print exhaustiveChange(42, [1,5,10,20,25])
```

```
[2, 0, 0, 2, 0]
```

Other Tricks?

- A branch and bound algorithm

```
def branchAndBoundChange(amount, denominations):
    bestN = amount
    count = [0 for i in xrange(len(denominations))]
    while True:
        for i, coinValue in enumerate(denominations):
            count[i] += 1
            if (count[i]*coinValue < amount):
                break
            count[i] = 0
        n = sum(count)
        if n == 0:
            break
        if (n > bestN):
            continue
        value = sum([count[i]*denominations[i] for i in xrange(len(denominations))])
        if (value == amount):
            if (n < bestN):
                solution = [count[i] for i in xrange(len(denominations))]
                bestN = n
    return solution

print branchAndBoundChange(42, [1,5,10,20,25])
```

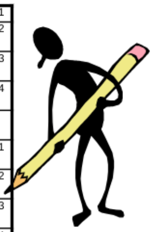
```
[2, 0, 0, 2, 0]
```

- Correct, and works well for most cases, but might be as slow as an exhaustive search for some inputs.

Is there another Approach?

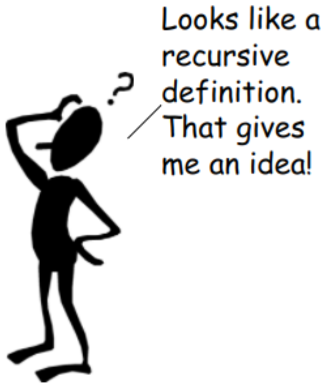
- Tabulating Answers
  - If it is costly to compute the answer for a given input, then there may be advantages to caching the result of previous calculations in a table
  - This trades-off time-complexity for space
  - How could we fill in the table in the first place?
  - Run our best correct algorithm
  - Can the table itself be used to speed up the process?

Amt	25	20	10	5	1	Amt	25	20	10	5	1
1¢					1	42¢		2			2
2¢					2	43¢		2			3
3¢					3	44¢		2			4
4¢					4	45¢		2		1	
5¢				1		46¢		2		1	1
6¢				1	1	47¢		2		1	2
7¢				1	2	48¢		2		1	3
8¢				1	3	49¢		2		1	4
9¢				1	4	50¢	2				
10¢			1			51¢	2				1
11¢			1		1	52¢	2				2



Solutions using a Table

- Suppose you are asked to fill-in the unknown table entry for 67¢
- It must differ from previous known optimal result by at most one coin…
- So what are the possibilities?
  - BestChange(67¢) = 25¢ + BestChange(42¢), or
  - BestChange(67¢) = 20¢ + BestChange(47¢), or
  - BestChange(67¢) = 10¢ + BestChange(57¢), or
  - BestChange(67¢) = 5¢ + BestChange(62¢), or
  - BestChange(67¢) = 1¢ + BestChange(66¢)



A Recursive Coin-Change Algorithm

```
def RecursiveChange(M, c):
    if (M == 0):
        return [0 for i in xrange(len(c))]
    smallestNumberOfCoins = M+1
    for i in xrange(len(c)):
        if (M >= c[i]):
            thisChange = RecursiveChange(M - c[i], c)
            thisChange[i] += 1
            if (sum(thisChange) < smallestNumberOfCoins):
                bestChange = thisChange
                smallestNumberOfCoins = sum(thisChange)
    return bestChange

print RecursiveChange(42, [1,5,10,20,25])
```

```
[2, 0, 0, 2, 0]
```

- The only problem is... it is too slow
- Let’s see why...

Recursion Recalculations

- Recursion often results in many redundant calls
- Even after only two levels of recursion 6 different change values are repeated multiple times
- How can we avoid this repetition?
- Cache precomputed results in a table!

Change(40) = 25 + Change(15)
25 + 10 + Change(5)
25 + 5 + Change(10)
20 + Change(20)
20 + 20 + Change(0)
20 + 10 + Change(10)
20 + 5 + Change(15)
10 + Change(30)
10 + 25 + Change(5)
10 + 20 + Change(10)
10 + 10 + Change(20)
10 + 5 + Change(25)
5 + Change(35)
5 + 25 + Change(15)
5 + 20 + Change(10)
5 + 10 + Change(25)
5 + 5 + Change(30)

Back to Table Evaluation

- When do we fill in the values of the table?
- We could do it lazily as needed… as each call to BestChange() progresses from M down to 1
- Or we could do it from the bottom-up – tabulating all values from 1 up to M
- Thus, instead of just trying to find the minimal number of coins to change M cents, we attempt the solve the superficially harder problem of solving for the optimal change for all values from 1 to M

1¢ = [0,0,0,0,1]
2¢ = [0,0,0,0,2]
3¢ = [0,0,0,0,3]
. . .
M¢ = [?,?,?,?,?]

Change via Dynamic Programming

```python
def DPChange(M, c):
    change = [[0 for i in xrange(len(c))]]
    for m in xrange(1,M+1):
        bestNumCoins = m+1
        for i in xrange(len(c)):
            if (m >= c[i]):
                thisChange = [x for x in change[m - c[i]]]
                thisChange[i] += 1
                if (sum(thisChange) < bestNumCoins):
                    change[m:m] = [thisChange]
                    bestNumCoins = sum(thisChange)
    return change[M]

print DPChange(42, [1,5,10,20,25])
```

```
[2, 0, 0, 2, 0]
```

- Recall, BruteForceChange( ) was O(M^d)
- DPChange( ) is O(Md)

Dynamic Programming

- Dynamic Programming is a general technique for computing recurrence relations efficiently by storing partial or intermediate results
- Three keys to constructing a dynamic programming solution:

    1. Formulate the answer as a recurrence relation
    2. Consider all instances of the recurrence at each step
    3. Order evaluations so you will always have precomputed the needed partial results

- We'll see it again, and again

Next Time

- Back to sequence alignment
- Another algorithm design approach.. Divide and Conquer